

Finite-Time Euler singularities: A Lagrangian perspective

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We address the question whether a singularity in a three-dimensional incompressible inviscid fluid flow can occur in finite time. Analytical considerations and numerical simulations suggest high-symmetry flows being a promising candidate for a finite-time blowup. Utilizing Lagrangian and geometric non-blowup criteria, we present numerical evidence against the formation of a finite-time singularity for the high-symmetry vortex dodecapole initial condition. We use data obtained from high resolution adaptively refined numerical simulations and inject Lagrangian tracer particles to monitor geometric properties of vortex line segments. We then verify the assumptions made by analytical non-blowup criteria introduced by Deng et. al [Commun. PDE **31** (2006)] connecting vortex line geometry (curvature, spreading) to velocity increase to rule out singular behavior.

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Introduction The incompressible Euler equations in three dimensions are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0. \quad (1)$$

Existence and uniqueness of its solutions for all times are unknown. Together with its prominent brother, the incompressible Navier-Stokes equations, these equations have withstood the minds of mathematicians and physicists for centuries. While the latter are regarded as “Millennium Prize Problem” by the Clay Mathematics Institute [1], the ignorance regarding existence of global solutions is even larger for the inviscid case: The notion of weak solutions, which is well established for the Navier-Stokes equations since Leray [2], is unknown for the three-dimensional Euler equations.

As a now classical result, the blowup criterion of Beale et al. [3] (BKM) connects the existence of solutions for the incompressible Euler equations in three dimensions to the critical accumulation of vorticity. It has been tried in the past to construct explicit initial conditions to obtain numerical evidence for or against a finite-time singularity via BKM, with surprisingly inconsistent results [4, 5]. The major reason for this ambiguity is the critical dependence on extrapolation, which renders the identification of singular versus near-singular behavior next to impossible by numerical means. The hopes are high that the situation is less vague when considering geometric analysis of the flow [6–9]. In this letter, we present the application of such geometric criteria to numerical data to sharpen the distinction between singular and near-singular flow evolution.

The Letter is organized as follows: we first review the notion of geometric non-blowup criteria and state the considered criteria and their interpretation. We then briefly name the computational setup and implementation details of our numerical scheme to obtain adaptively refined data of up to 8192^3 mesh points. Using Lagrangian tracers and diagnostics for vortex line geometry,

we analyze the simulation data to conclude a non-blowup for the considered initial conditions. A conclusion and outlook summarize the Letter.

Geometric non-blowup criteria. Historically, non-blowup criteria for the incompressible Euler equations commonly focus on global features of the flow, such as norms of the velocity or the vorticity fields. This comes at the disadvantage of neglecting the structures and physical mechanisms of the flow evolution. A strategy to overcome such shortcomings was established by focusing more on geometrical properties and flow structures, such as vortex tubes or vortex lines. Starting with the works of Constantin [10], Constantin et al. [6], and Cordoba and Fefferman [7], some of these geometric criteria (e.g. [8, 11, 12]) have reached a phase where they allow direct verification of their assumptions with the help of numerical simulations.

Common to geometric criteria is the notion of vortex lines, defined as integral curves along the vorticity direction field $\boldsymbol{\xi}$. They are transported with the flow, i.e. two points \mathbf{x} and \mathbf{y} on the same vortex line $c(s)$ stay on the same vortex line indefinitely. As simple consequence of the solenoidality of $\boldsymbol{\omega}$ and the BKM theorem, one gets:

Deng-Hou-Yu theorem 1: *Let $\mathbf{x}(t)$ be a family of points such that for some $c_0 > 0$ it holds $|\boldsymbol{\omega}(\mathbf{x}(t), t)| > c_0 \Omega(t)$. Assume that for all $t \in [0, T)$ there is another point $\mathbf{y}(t)$ on the same vortex line as $\mathbf{x}(t)$, such that the direction of vorticity $\boldsymbol{\xi}(\mathbf{x}, t) = \boldsymbol{\omega}(\mathbf{x}, t)/|\boldsymbol{\omega}(\mathbf{x}, t)|$ along the vortex line $c(s)$ between $\mathbf{x}(t)$ and $\mathbf{y}(t)$ is well-defined. If we further assume that*

$$\left| \int_{\mathbf{x}(t)}^{\mathbf{y}(t)} (\nabla \cdot \boldsymbol{\xi})(c(s), t) ds \right| \leq C \quad (2)$$

for some absolute constant C , and

$$\int_0^T |\boldsymbol{\omega}(\mathbf{y}(t), t)| dt < \infty, \quad (3)$$

then there will be no blowup up to time T .

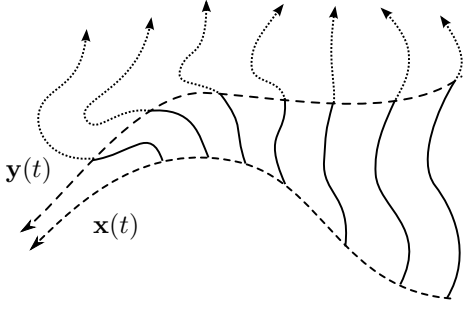


FIG. 1: For the position $\mathbf{x}(t)$ of maximum vorticity, choose $\mathbf{y}(t)$ such that $\left| \int_{\mathbf{x}(t)}^{\mathbf{y}(t)} (\nabla \cdot \boldsymbol{\xi})(c(s), t) ds \right| = C$. For a point-wise singularity, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ must collapse in finite time.

This criterion can readily be applied to numerical simulations. On the other hand, the same theorem may be interpreted in a different way to distinguish between a point-wise blowup and the blowup of a complete vortex line segment, as sketched in Fig. 1: At each instance in time, identify the point of maximum vorticity as $\mathbf{x}(t)$. Now define $\mathbf{y}(t)$ such that $\left| \int_{\mathbf{x}(t)}^{\mathbf{y}(t)} (\nabla \cdot \boldsymbol{\xi})(c(s), t) ds \right| = C$ for a constant threshold C . If a singularity occurs, then either $\mathbf{y}(t)$ approaches $\mathbf{x}(t)$ (point-wise blowup), or, if the distance between $\mathbf{x}(t)$ and $\mathbf{y}(t)$ stays finite, the complete vortex line segment between $\mathbf{x}(t)$ and $\mathbf{y}(t)$ exhibits critical growth.

Results obtained with this method can be further improved by considering the Lagrangian evolution of vortex line segments L_t in time. The geometric equivalent of the vortex stretching term is the increase in length for a Lagrangian vortex line segment. It is possible to quantify this stretching and establish a sound connection to the vorticity dynamics of the flow.

Denote with $l(t)$ the length of a vortex line segment L_t at time t and define with $\Omega_L(t) := \|\boldsymbol{\omega}(\cdot, t)\|_{L^\infty(L_t)}$ the maximum vorticity on the vortex line segment. Furthermore, let $M(t) := \max(\|\nabla \cdot \boldsymbol{\xi}\|_{L^\infty(L_t)}, \|\kappa\|_{L^\infty(L_t)})$ be the quantity of vortex line convergence $\nabla \cdot \boldsymbol{\xi}$ and vortex line curvature κ , and define $\lambda(L_t) := M(t)l(t)$. It then holds (compare Fig. 2):

Deng-Hou-Yu theorem 2: Assume there is a family of vortex line segments L_t and $T_0 \in [0, T)$, such that $L_{t_2} \subseteq \mathbf{X}(L_{t_1}, t_1, t_2)$ for all $T_0 < t_1 < t_2 < T$. We also assume that $\Omega(t)$ is monotonically increasing and $\|\boldsymbol{\omega}(t)\|_{L^\infty(L_t)} \geq c_0 \Omega(t)$ for some $c_0 > 0$ when t is sufficiently close to T . Furthermore, we assume that

$$(i) \quad U_\xi(t) + U_n(t)\lambda(L_t) \lesssim (T-t)^{-A} \text{ for } A \in (0, 1)$$

$$(ii) \quad \lambda(L_t) \leq C_0,$$

$$(iii) \quad l(t) \gtrsim (T-t)^B \text{ for some } B < 1 - A.$$

Then there will be no blowup in the 3D incompressible Euler flow up to time T .

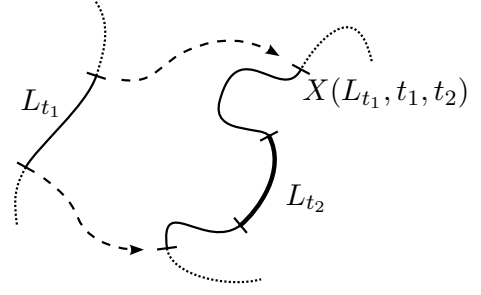


FIG. 2: Lagrangian evolution of a vortex line segment L_t in the context of theorem 2. For every $t_2 > t_1$, choose L_{t_2} such that it is a subset of $\mathbf{X}(L_{t_1}, t_1, t_2)$.

Here, $a(t) \lesssim b(t)$ means that there exists a constant $c \in \mathbb{R}$ such that $|a(t)| < c|b(t)|$. The velocity components are defined as $U_\xi(t) := \max_{\mathbf{x}, \mathbf{y} \in L_t} |(\mathbf{u} \cdot \boldsymbol{\xi})(\mathbf{x}, t) - (\mathbf{u} \cdot \boldsymbol{\xi})(\mathbf{y}, t)|$ and $U_n(t) := \max_{L_t} |\mathbf{u} \cdot \mathbf{n}|$. The proof is given in [8].

All information about the geometric properties of the vortex line segment under consideration is encoded in $\lambda(L_t)$, characterizing the geometric “tame”ness of the vortex line filament. As depicted in Fig. 3, a vortex line segment has a huge $\lambda(L_t)$, if its maximum curvature is large, relative to its length (the segment is “kinked” instead of “curved”), or if the surrounding vortex lines collapse to the considered segment in at least a point (the surrounding is “tightening” instead of “parallel”). This quantifies the constricted notion of “relatively straight” and “smoothly directed” given in [6] in a sharper way. The process of keeping $\lambda(L_t)$ bounded by some constant C_0 translates to words as the process of “zooming in” to the location of maximum vorticity in order to keep the considered vortex line segment relatively straight in comparison to its length. The assumed accompanying collapse in length to keep $\lambda(L_t)$ bounded is then linked in its growth rate to the blowup of the velocity components.

Despite high hopes from an analytical point of view that these considerations will shed light on the true nature of vorticity accumulation, numerical results observ-

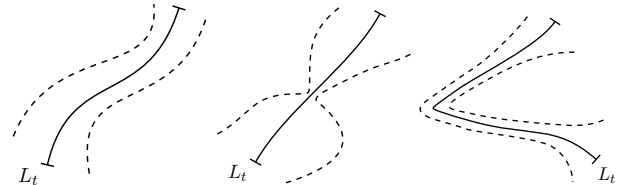


FIG. 3: Characterizing vortex line geometry in terms of $\lambda(L_t)$. A slightly curved vortex line with approximately parallel neighboring vortex lines (**left**) exhibits small $\lambda(L_t)$. Vortex lines with tightening neighboring vortex lines (**center**) or vortex lines with high curvature, in comparison to their length (**right**) have large $\lambda(L_t)$.

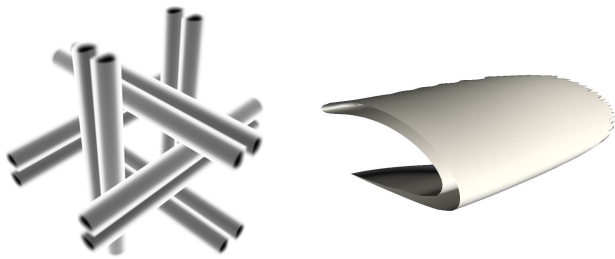


FIG. 4: Left: Volume plot of the vorticity for the vortex dodecapole initial conditions: Six pairs of anti-parallel vortex tubes. Right: Isosurface of the vorticity for a single tube at a late time.

ing geometrical properties of Lagrangian vortex filaments are scarce. This is primarily due to the fact that Eulerian quantities such as $\Omega(t)$ are readily trackable in post-processing, while monitoring the Lagrangian evolution requires additional computational effort. On top of that, the geometry of integral curves at an instance in time, though in principle computable in post-processing, as well as derived quantities such as their convergence and curvature, are quite inaccessible in comparison to simple Eulerian criteria.

Numerical Experiment Vorticity-strain coupling is the favored mechanism for the formation of a finite-time Euler singularity. Since it is well established that this process is inherently unstable for turbulent flows, it seems natural to search for techniques to artificially keep the coupling existent. One such technique is the introduction of symmetries to the flow. Early examples such as the Taylor-Green vortex [13] or Kerr’s initial conditions [4, 5] are employing such symmetries. Yet, as pointed out by Pelz [14], for a single vortex tube to exhibit critical vorticity-strain coupling, its curvature in the symmetry plane has to blow up alongside the axial strain. On the other hand, increasing axial strain diminishes the curvature of the critical vortex line. These counteracting processes constitute an intrinsic resistance of a single vortex line to locally “self-stretch” in a critical way. The same argument holds for a pair of anti-parallel vortex tubes. A popular way to counter this effect is to induce the axial strain by neighboring tubes instead of relying on a sufficiently large kink. This was accomplished, as suggested in [15], by introducing additional rotational symmetries to the flow, arriving at the vortex dodecapole initial condition pictured in Fig. 4 (left). These initial conditions are recognized as a promising candidate for the formation of a finite-time singularity. As additional benefit, the high symmetry introduces huge savings in computational effort and memory requirements. An isosurface of the vorticity of one of the vortex tubes at a late time is depicted in 4 (right), showing the typical roll-up of the vortex sheet in the critical region.

Resolution is paramount for a reliable statement on

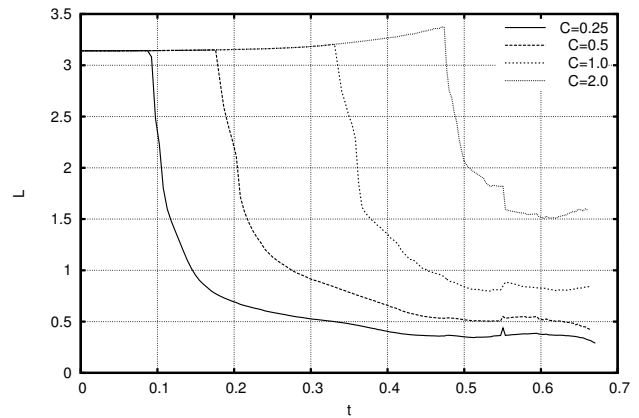


FIG. 5: Length of the vortex lines starting at position \mathbf{x} of maximum vorticity for different constants $C = \int_x^y \nabla \cdot \xi ds$.

possible singular behavior of the Euler equations. We use adaptively refined meshes provided by the framework *racoona* [16] to reach effective resolutions of up to 8192^3 mesh points, scaling close to optimal on up to 10^5 cores on massively parallel machines. The numerical scheme consists of a strong stability preserving third order Runge-Kutta [17] time integrator combined with a third order shock-capturing CWENO scheme [18] to reduce oscillations in the presence of strong gradients. The integrated equation is the vorticity formulation of the Euler equations, employing a vector potential formulation $\Delta \mathbf{A} = -\boldsymbol{\omega}$ with $\mathbf{u} = \nabla \times \mathbf{A}$ to ensure solenoidality of the vorticity vector field $\boldsymbol{\omega}$. The associated Poisson equation is solved with a second order parallel and adaptive multigrid algorithm. Interpolation on the coarse-fine interfaces is done in $\boldsymbol{\omega}$ to ensure the highest possible accuracy when applying the aforementioned blowup-criteria. Passive tracer particles are injected into the flow for the tracking of Lagrangian vortex line segments. The above third order Runge-Kutta is also used for the time integration of the tracer particles and the space integration of vortex lines. Details of the numerical scheme, regarding its implementation, adaptivity, parallelization, and diagnostics will be presented elsewhere.

The numerical experiment now consists of two parts: First, utilizing the method outlined above, we analyze the nature of the possible singularity in terms of locality (point-wise versus filament). Then, by means of the second theorem, we conclude from the scaling of length and velocity components of the critical Lagrangian vortex filament, if the observed behavior is singular at all.

Results For the first part, the constant C of theorem 1 is chosen in a reasonable way to achieve a length of the vortex line segment that fits into the computational domain in the beginning of the simulation, but is still well resolved at the chosen resolution at later times. Hence, the whole vortex line segment is resolved reliably throughout the simulation.

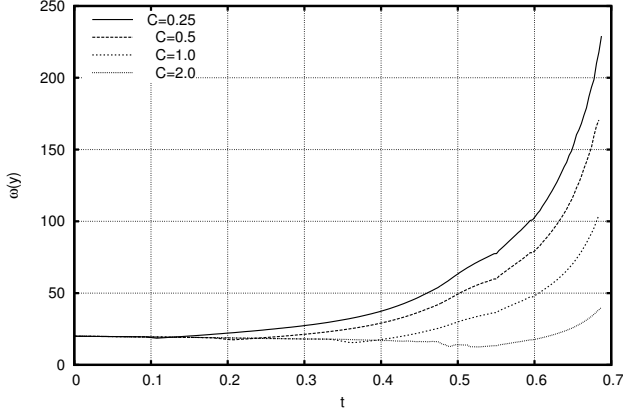


FIG. 6: Vorticity at the endpoint y of the considered vortex line segments. Once satiated, the growth rate is the same for all y .

The results for the vortex dodecapole are presented in figure 5 for different constants $C \in \{0.25, 0.5, 1, 2\}$. Initially, the vortex line segments do not accumulate enough $\nabla \cdot \xi$, so that the length is bounded by the size of the computational domain ($\mathbf{x} \in [0, \pi]^3$). At some point, depending on the value of C , the threshold is reached and the length of the vortex line segment decreases. Yet, for all considered cases of C , the length does not collapse to a point, but saturates at early times without approaching $l(t) = 0$. This behavior appears to be stable up to the latest time of the simulation. The final length of the vortex line segments is at least $l = 0.3$ for the smallest case of C ($C = 0.25$), which is still well resolved with at least $200 \Delta x$ for the simulation with 4096^3 grid points. This result, therefore, is a numerical evidence against a point-wise blowup for the vortex dodecapole class of initial conditions. This is in concordance with the estimate in [8].

Yet, monitoring the development of $\omega(\mathbf{y}(t), t)$, as shown in figure 6, yields a similar growth rate for the accumulation of vorticity at the endpoint $\mathbf{y}(t)$ as for the beginning of the vortex line segment $\mathbf{x}(t)$. This is hardly surprising, since by construction a constant value for C directly links the growth rates of $|\omega(\mathbf{x}(t), t)|$ to $|\omega(\mathbf{y}(t), t)|$. Nevertheless, a numerical verification of this analytic equality may be seen as a confirmation that the observed growth rate of $|\omega(\mathbf{x}(t), t)|$ is by no means a numerical artifact in an isolated small area, but is reproduced at points far away from the critical region, which appear to be well-behaved at first view. The possibly critical growth in the perspective of BKM is, thus, confirmed by the global flow.

Furthermore, since for a large portion of the simulation the distance $l(t)$ is approximately constant, this could be interpreted as an evidence for the existence of a non-vanishing vortex line segment that blows up in every point. Thus, contradicting the estimation of [8], the pos-

sibility of a blowup of the vortex dodecapole flow is not excluded by theorem 1. The scenario of a collapse to a single point, on the other hand, is clearly conflicting the numerical evidence.

For the second part of the numerical experiment, the above mentioned theorem 2 is verified numerically to rule out a singularity in finite time. The strategy is as follows:

- Identify the Lagrangian fluid element α , which will contain the maximum of vorticity at the latest time of the simulation, $\Omega(t) \approx |\omega(\mathbf{X}(\alpha, t), t)|$. Numerically this procedure is implemented by carrying out a precursory identical simulation with a huge number of tracer particles (≈ 1 million) randomly distributed across the domain. Particles that accumulate huge amounts of vorticity are selected for the production run.
- In a subsequent computation, at each instance in time, start a vortex line integration at $\mathbf{X}(\alpha, t)$ along the vorticity direction field. Monitor the maximum curvature $\|\kappa\|_{L^\infty(L_t)}$ and the maximum vortex line convergence $\|\nabla \cdot \xi\|_{L^\infty(L_t)}$ during the integration and calculate $\lambda(t)$. Stop the integration, as soon as $\lambda(t)$ reaches a fixed, arbitrary constant C . This defines L_t .
- For this vortex line segment L_t , calculate the length $l(t)$, and the velocity components U_n and U_ξ . From the collapse of the length $l(t)$ approximate the exponent B . This in turn provides the critical growth exponent A for the velocity variables, $A_{\text{crit}} = 1 - B$.
- Compare the increase in U_n and U_ξ to $1/(T-t)^{A_{\text{crit}}}$ to distinguish between critical and sub-critical growth of velocity.

This can be interpreted rather intuitively. By prescribing an arbitrarily fixed $\lambda(t)$, the vortex line segment is kept relatively geometrically uncritical, as the length-scale is always adjusted accordingly. This process of “zooming in” just enough to retain the geometric “criticalness” prescribes the rate of collapse to a point, at least in the direction of the vortex line. All that is left to check is whether the velocity growth in the immediate surrounding is fast enough to be compatible with a finite-time singularity.

Figure 7 shows the results for the vortex dodecapole initial conditions. Pictured is the length of the vortex line segment for the tracer that is arriving at a position of very huge vorticity at late stages of the simulation. The subplot depicts the long-term behavior of the particle entering the critical region, while the final stage of length decrease is magnified. The decrease in length does not agree with a collapse in final time, but instead the shrinkage of the segment decelerates clearly in time. This contradicts a scaling in time proportional to $(T-t)^B$ for

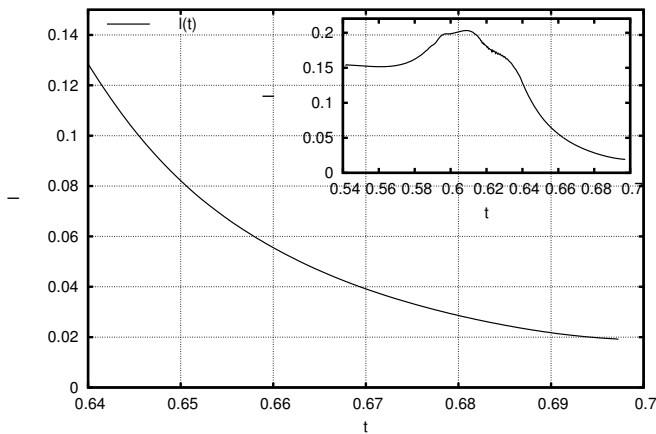


FIG. 7: Evolution of the length $l(t)$ of the critical vortex filament L_t for different Lagrangian fluid elements. The length does not decrease as $(T - t)^B$ for any $B < 1$, which would be faster than linear. The Lagrangian collapse of the vortex segment is decelerating instead.

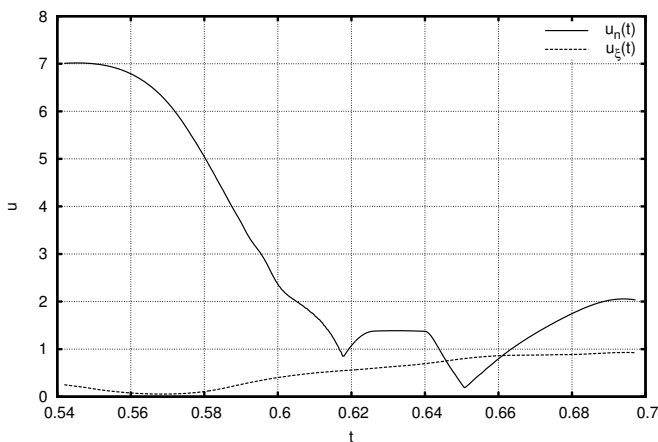


FIG. 8: Evolution of the quantities U_n and U_ξ in time. Both quantities do not exhibit a critical growth of $1/(T - t)$.

any $0 < B \leq 1$, which would be faster than (or, in the limiting case, equal to) linear.

It could furthermore be argued that the limit $B \rightarrow 0$ is hard to exclude, since the drop in length would be virtually instantaneous in time, with a close to constant scaling before. In this limit, the quantities U_n and U_ξ would have to grow roughly as $1/(T - t)$ to still allow formation of a finite-time singularity. Figure 8 shows the observed behavior of U_n and U_ξ in time. Both show no signs of critical accumulation, in particular not like $1/(T - t)$ in time. Thus, the assumptions of theorem 2 are well met. These results therefore pose a strong evidence against a finite-time singularity for the class of vortex dodecapole initial conditions.

Conclusions and Outlook In this Letter we studied the question whether a singularity in a three-dimensional incompressible inviscid fluid flow can occur in finite time. Using massively parallel high-resolution adaptive simulations and applying Lagrangian and geometric diagnostics to the flow evolution, we are able to probe the behavior of the critical vortex filament. Our findings pose a strong numerical evidence against a finite-time singularity for the vortex dodecapole initial conditions, based on analytical criteria connecting velocity scaling to vortex line segment geometry. In principle, the presented method could easily be applied to different classes of initial conditions.

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